Exact solution of the Ising model on checkerboard fractals in an external magnetic field

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In this paper we use a diagrammatic technique to determine the exact recursion relations for the partition function of the Ising model in an external magnetic field, situated on the first two members of the checkerboard family of fractal lattices embedded in two dimensions. This represents the first exact general solution of this model for the case of nonzero field. The closed-form expression for the partition function is obtained for the first member in the zero-field limit and the nonzero-field recursion relations prove sufficient for exact evaluation of the response functions. We also calculate the temperature dependence of the specific heat and susceptibility. [S1063-651X(98)04106-3]

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I. INTRODUCTION

In spite of the fact that finitely ramified fractals present one of the rare cases where the Ising model is (at least in principle) exactly solvable [1–4], for the case of the nonzero external magnetic field to date *no such closed-form solutions exist*. An exact closed-form solution for the partition function of the Ising model on the first member of the checkerboard (CB) fractal family (with generator side length $b=3$) in zero field, in two and three dimensions, was recently derived by Yang [5] using graph theory, but the nonzero-field case has still defied solution. On the other hand, the nonzero-field case turns out to be amenable to exact renormalization-group (RG) analysis and to date exact RG recursion relations have been established [6,7] for the members of the Sierpínski gasket fractal family with generator side length $b=2,3,\ldots,8$.

In this work we establish the exact recursion relations for the partition function of the Ising model on the first two members of the CB fractal family (with generator side lengths $b=3$ and $b=5$), embedded in two dimensions, which prove sufficient for exact evaluation of the response functions. In the special case of zero external magnetic field, the recursion relations yield a closed-form expression identical to that of Yang [5]. In the general case of nonzero field the closed-form expression for the partition function still does not seem to be tractable, but the fact that the field and temperature dependence of the response functions can be numerically obtained from the recursion relations with *arbitrary precision* renders this approach equivalent to having the closed-form solution in the field.

To obtain the recursion relations for the partition function, we extend the diagrammatic procedure developed in a previous work [7] by introducing auxiliary steps in the recursion process. The exact recursion relations are obtained manually for the $b=3$ CB fractal and for the $b=5$ case we use a devoted symbolic computational algorithm. From these relations we then find the recursion relations for the temperature and field derivatives of the partition function, which can be iterated for arbitrary parameter values to yield the response functions with any desired precision.

In the next section we present the modification of the diagrammatic technique developed in Ref. [7] and derive the recursion relations for the partition function of the first member of the CB fractal family in nonzero external magnetic field. In Sec. III we discuss the symbolic-numerical approach used to find the recursion relations for the second member of the family and in Sec. IV we present the results obtained for the thermodynamic response functions. Finally, in Sec. V we draw the conclusions.

II. EXACT RECURSION RELATIONS FOR THE PARTITION FUNCTION OF THE CB FRACTAL WITH BASE $b=3$

In this section we extend the diagrammatic technique introduced in Ref. [7] for more convenient application to the specific case of members of the CB family in an external magnetic field. The recursion relations will prove to be exact in all stages of construction of the fractal, starting from the generator up to the thermodynamic limit.

We consider Ising spins located on vertices of CB fractal lattices, interacting through the Hamiltonian

$$\mathcal{H}=-J\sum_{\langle\mathrm{NN}\rangle} S_iS_j-H\sum_i S_i, \tag{1}$$

where J is the nearest-neighbor exchange coupling, $S_i=\pm1$ is the Ising spin variable at the lattice site i, H is the external magnetic field, and $\langle\mathrm{NN}\rangle$ denotes summation over the nearest-neighbor pairs. The first two stages of construction of the first two members of the CB family are shown in

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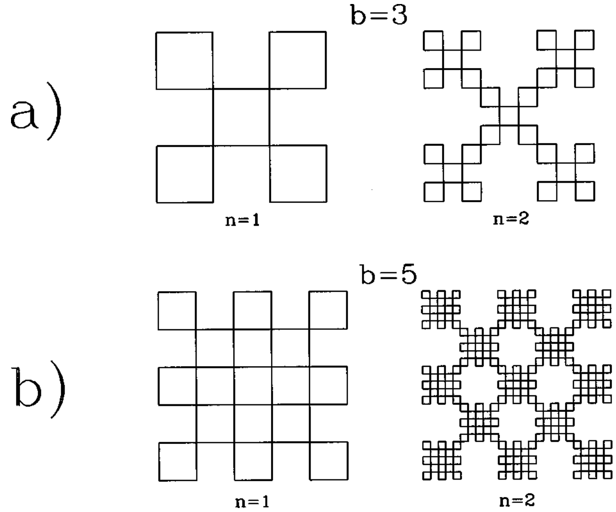

FIG. 1. First two stages of construction ($n=1$ and $n=2$) of the first two members of the checkerboard fractal family with generator side length (a) $b=3$ and (b) $b=5$. Fractal lattices are obtained in the limit $n\rightarrow\infty$.

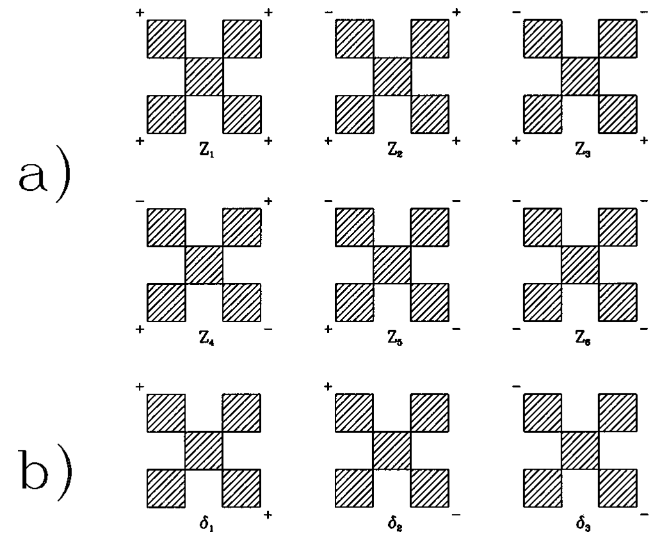

FIG. 2. (a) Spin configurations corresponding to the six independent partial partition functions Z_1, Z_2, Z_3, Z_4, Z_5, and Z_6, with $\{++++\}$, $\{+++-\}$, $\{++--\}$, $\{+-+-\}$, $\{+---\}$, and $\{----\}$ configurations of the vertex spins. The shaded areas indicate the existent substructure, with undetermined orientation of constituent spins. (b) Spin configurations corresponding to the three independent auxiliary partial partition functions δ_1, δ_2, and δ_3, corresponding to $\{++\}$, $\{+-\}$, and $\{--\}$ configurations of the diagonal spins.

Fig. 1. The lattice at stage n is obtained by joining n_c structures of the $(n-1)$st stage, *exclusively* by their vertices, with $n_c=5$ for the first member ($b=3$) and $n_c=13$ for the second member ($b=5$), n_c being independent of the stage of construction n.

At any stage of construction of a CB fractal, one can define 16 partial partition functions $\{Z_i, i=1,\dots,16\}$ that correspond to possible configurations of the four vertex spins, with a summation performed over all the other spins. From the symmetry of the lattice, however, it follows that at each stage there are only six independent partial partition functions Z_1, Z_2, Z_3, Z_4, Z_5, and Z_6, corresponding to $\{++++\}$, $\{+++-\}$, $\{++--\}$, $\{+-+-\}$, $\{+---\}$, and $\{----\}$ configurations of the vertex spins, respectively. Spin configurations corresponding to independent partial partition functions are schematically shown in Fig. 2(a). If the recursion relations between the sets of partial partition functions (corresponding to any two consecutive stages of construction of the lattice) are established, the partition function at arbitrary stage of construction of the lattice can be found as

$$Z=Z_1+4Z_2+4Z_3+2Z_4+4Z_5+Z_6 \qquad (2)$$

and the thermodynamic response functions can be obtained as temperature and field derivatives of Z. The recursion relations for the partial partition functions could in principle also be used to obtain the real-space RG relations for the interaction parameters as well as the closed-form expression for the partition function.

To construct the recursion relations for the partial partition functions we have introduced in Ref. [7] a convenient diagrammatic technique, based on inspection of graphs that correspond to two consecutive stages of construction. With each of the six partial partition functions at stage n we associate all graphs with vertex spins fixed in the corresponding configuration, for all possible configurations of the interior spins [we term interior spins those spins that are not at vertices at the nth stage of construction, but are the vertex spins of the constituent $(n-1)$st stage structures]. Each of the graphs corresponds to a single term in the recursion relation, which is obtained as the product of partial partition functions of the constituent $(n-1)$st stage structures and an exponential multiplicative factor

$$\exp\left[-\beta H\sum_{i=1}^{N_i} S_i(r_i-1)\right],$$

where the sum is performed over the N_i interior spins, r_i is the number of $(n-1)$st stage structures sharing in the nth stage the ith interior site, and $\beta=1/k_BT$ is the standard notation for the reciprocal of the product of the Boltzmann constant and temperature. This factor accounts for the difference between the energy of the nth stage structure and the simple sum of energies of the $(n-1)$st stage structures. The problem with the checkerboard lattices is that already the first member with $b=3$ has $N_i=12$ interior spins, so that there are $2^{12}=4096$ graphs associated with each of the six recursion relations for the partial partition functions. This number of graphs cannot be handled manually and indeed, in Sec. III, we will discuss the symbolic algebraic algorithm used for the second member ($b=5$) with $N_i=32$ interior spins corresponding to $2^{32}=4\,294\,967\,296$ graphs for each of the six recursion relations. The number of necessary graphs is, however, drastically reduced if we take into account the fact that out of the 12 interior spins only four are shared by two $(n-1)$st stage substructures ($r_i=2$), while the other eight belong only to a single substructure ($r_i=1$). We can thus assume that the summation over these eight spins has already been performed and, in addition to the six partial partition functions $Z_1, Z_2, \dots, Z_6$, consider the auxiliary partial partition functions δ_1, δ_2, and δ_3 corresponding

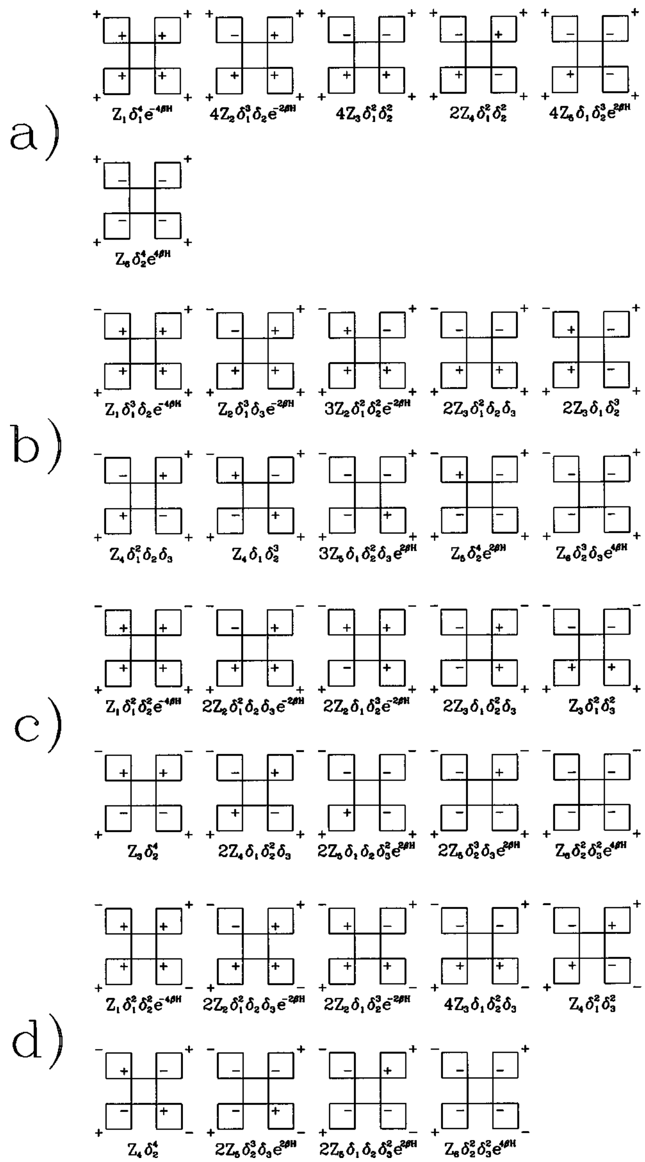

FIG. 3. Diagrams used for determining the recursion relations for the partial partition functions (a) Z_1, (b) Z_2, (c) Z_3, and (d) Z_4. In each case, only one representative member of a given symmetry class is shown, which is indicated by the integer multiplicative factor, each term being a product of one partial and four auxiliary partition functions. The remaining two recursion relations for Z_5 and Z_6 are obtained from those for Z_2 and Z_1, respectively, by substituting $Z_1 \leftrightarrow Z_6$, $Z_2 \leftrightarrow Z_5$, $\delta_1 \leftrightarrow \delta_3$, and $H \leftrightarrow -H$.

to $\{++\}$, $\{+-\}$, and $\{--\}$ configurations of spins on the diagonal of the lattice, schematically shown in Fig. 2(b). If these functions are known for the $(n-1)$st stage together with the six partial partition functions, there are only four interior spins left to be considered in the construction of the nth stage. The total number of graphs in the case $b=3$ is thus reduced to just $2^4=16$ graphs associated with each of the six recursion relations for the partial partition functions and in Fig. 3 these graphs are represented for partial partition functions Z_i, $i=1,\ldots,4$.

Each graph now corresponds to the product of the partial partition function Z_i of the central $(n-1)$st stage structure, the auxiliary partial partition functions δ_i of the peripheral four $(n-1)$st stage structures, and the exponential field-dependent term

$$\exp\left(-\beta H \sum_{i=1}^{4} S_i\right).$$

By inspection of Fig. 3 we find the following recursion relations between *any two* consecutive stages of construction:

$$Z_1' = Z_1 \delta_1^4 e^{-4\beta H} + 4 Z_2 \delta_1^3 \delta_2 e^{-2\beta H} + 4 Z_3 \delta_1^2 \delta_2^2 + 2 Z_4 \delta_1^2 \delta_2^2 + 4 Z_5 \delta_1 \delta_2^3 e^{2\beta H} + Z_6 \delta_2^4 e^{4\beta H}, \tag{3a}$$

$$Z_2' = Z_1 \delta_1^3 \delta_2 e^{-4\beta H} + Z_2(\delta_1^3 \delta_3 + 3\delta_1^2 \delta_2^2) e^{-2\beta H} + 2 Z_3(\delta_1^2 \delta_2 \delta_3 + \delta_1 \delta_2^3) + Z_4(\delta_1^2 \delta_2 \delta_3 + \delta_1 \delta_2^3) + Z_5(3\delta_1 \delta_2^2 \delta_3 + \delta_2^4) e^{2\beta H} + Z_6 \delta_2^3 \delta_3 e^{4\beta H}, \tag{3b}$$

$$Z_3' = Z_1 \delta_1^2 \delta_2^2 e^{-4\beta H} + 2 Z_2(\delta_1^2 \delta_2 \delta_3 + \delta_1 \delta_2^3) e^{-2\beta H} + Z_3(2\delta_1 \delta_2^2 \delta_3 + \delta_1^2 \delta_3^2 + \delta_2^4) + 2 Z_4 \delta_1 \delta_2^2 \delta_3 + 2 Z_5(\delta_1 \delta_2 \delta_3^2 + \delta_2^3 \delta_3) e^{2\beta H} + Z_6 \delta_2^2 \delta_3^2 e^{4\beta H}, \tag{3c}$$

$$Z_4' = Z_1 \delta_1^2 \delta_2^2 e^{-4\beta H} + 2 Z_2(\delta_1^2 \delta_2 \delta_3 + \delta_1 \delta_2^3) e^{-2\beta H} + 4 Z_3 \delta_1 \delta_2^2 \delta_3 + Z_4(\delta_1^2 \delta_3^2 + \delta_2^4) + 2 Z_5(\delta_2^3 \delta_3 + \delta_1 \delta_2 \delta_3^2) e^{2\beta H} + Z_6 \delta_2^2 \delta_3^2 e^{4\beta H}. \tag{3d}$$

From the symmetry of the lattice it follows that the remaining two recursion relations for Z_5 and Z_6 are obtained from those for Z_2 and Z_1, respectively, by substituting $Z_1 \leftrightarrow Z_6$, $Z_2 \leftrightarrow Z_5$, $\delta_1 \leftrightarrow \delta_3$, and $H \leftrightarrow -H$, yielding

$$Z_5' = Z_1 \delta_1 \delta_2^3 e^{-4\beta H} + Z_2(3\delta_1 \delta_2^2 \delta_3 + \delta_2^4) e^{-2\beta H} + 2 Z_3(\delta_1 \delta_2 \delta_3^2 + \delta_2^3 \delta_3) + Z_4(\delta_1 \delta_2 \delta_3^2 + \delta_2^3 \delta_3) + Z_5(3\delta_2^2 \delta_3^2 + \delta_1 \delta_3^3) e^{2\beta H} + Z_6 \delta_2 \delta_3^3 e^{4\beta H}, \tag{3e}$$

$$Z_6' = Z_1 \delta_2^4 e^{-4\beta H} + 4 Z_2 \delta_2^3 \delta_3 e^{-2\beta H} + 4 Z_3 \delta_2^2 \delta_3^2 + 2 Z_4 \delta_2^2 \delta_3^2 + 4 Z_5 \delta_2 \delta_3^3 e^{2\beta H} + Z_6 \delta_3^4 e^{4\beta H}. \tag{3f}$$

Furthermore, from Fig. 2 it is easily deduced that the auxiliary partial partition functions $\{\delta_i,\ i=1,2,3\}$ can be expressed in terms of the partial partition functions $\{Z_i,\ i=1,\ldots,6\}$ as

$$\delta_1 = Z_1 + 2 Z_2 + Z_4, \tag{4a}$$

$$\delta_2 = Z_2 + 2 Z_3 + Z_5, \tag{4b}$$

$$\delta_3 = Z_4 + 2 Z_5 + Z_6. \tag{4c}$$

By substituting Eqs. (4) into Eqs. (3) explicit expressions for the recursion relations of the partial partition functions are obtained.

Finally, for the initial conditions of the recursion relations (3) one can choose the partial partition functions of a simple square (zeroth stage of construction of the fractal), given by

$$Z_1 = e^{4\beta J + 4\beta H}, \tag{5a}$$

$$Z_2 = e^{2\beta H}, \tag{5b}$$

$$Z_3=1, \tag{5c}$$

$$Z_4=e^{-4\beta J}, \tag{5d}$$

$$Z_5=e^{-2\beta H}, \tag{5e}$$

$$Z_6=e^{4\beta J-4\beta H}. \tag{5f}$$

In order to make contact with the conventional RG spin decimation approach, it seems worthwhile commenting on the equivalence of recursion relations (3) for the partial partition functions with the standard real-space RG recursion relations for the interaction parameters, as well their potential to yield a closed-form expression for the partition function.

To explicitly write the standard real-space RG equations for the closed set of interactions $\{K_i\}$, instead of Eq. (1) a Hamiltonian with six terms needs to be considered (otherwise new couplings are generated at initial RG steps). Expressions equivalent to Eqs. (5) would then provide the relation between the partial partition functions and the set of renormalized couplings (which represent the parameters of the standard real-space RG procedure) at an arbitrary stage of construction of the fractal. To obtain the standard RG recursion relations, one would further have to insert these equations into Eqs. (3) and (4) and solve for $\{K_i'$, $i=1,\ldots,6\}$ in terms of $\{K_i,\ i=1,\ldots,6\}$.

The closed-form expression for the partition function could also in principle be obtained from Eqs. (2)–(4). This becomes feasible in the zero-field case, where these equations can be used to obtain the recursion relation for the (full) partition function at two arbitrary steps of construction of the fractal, which is then iterated to yield a closed-form expression. That is, because of the symmetry in zero field there are only four independent partial partition functions Z_1, Z_2, Z_3, and Z_4 and two auxiliary partial partition functions δ_1 and δ_2. Substituting $Z_5=Z_2$, $Z_6=Z_1$, $H=0$, and $\delta_1=\delta_3$ into Eqs. (2)–(4) we obtain the recursion relations

$$Z_1'=Z_1(\delta_1^4+\delta_2^4)+4Z_2(\delta_1^3\delta_2+\delta_1\delta_2^3)+4Z_3\delta_1^2\delta_2^2+2Z_4\delta_1^2\delta_2^2, \tag{6a}$$

$$Z_2'=Z_1(\delta_1^3\delta_2+\delta_1\delta_2^3)+Z_2(\delta_1^4+6\delta_1^2\delta_2^2+\delta_2^4)+2Z_3(\delta_1^3\delta_2+\delta_1\delta_2^3)+Z_4(\delta_1^3\delta_2+\delta_1\delta_2^3), \tag{6b}$$

$$Z_3'=2Z_1\delta_1^2\delta_2^2+4Z_2(\delta_1^3\delta_2+\delta_1\delta_2^3)+Z_3(\delta_1^4+2\delta_1^2\delta_2^2+\delta_2^4)+2Z_4\delta_1^2\delta_2^2, \tag{6c}$$

$$Z_4'=2Z_1\delta_1^2\delta_2^2+4Z_2(\delta_1^3\delta_2+\delta_1\delta_2^3)+4Z_3\delta_1^2\delta_2^2+Z_4(\delta_1^4+\delta_2^4), \tag{6d}$$

and

$$\delta_1=Z_1+2Z_2+Z_4, \tag{7a}$$

$$\delta_2=2Z_2+2Z_3, \tag{7b}$$

with

$$Z=2Z_1+8Z_2+4Z_3+2Z_4 \tag{8}$$

or, equivalently,

$$Z=2(\delta_1+\delta_2). \tag{9}$$

Substituting Eqs. (7) into Eqs. (6) and then the result into Eq. (8), we find the following extremely simple recursion relation for partition function at two arbitrary successive steps of construction:

$$Z'=2(Z/2)^5, \tag{10}$$

where we have used MAPLE V algebraic manipulation software package to perform this straightforward but tedious task. Starting from the partition function Z_0 of a single square (zeroth stage of construction of the fractal) and iterating Eq. (10) n times, for the partition function at the nth stage of construction it is easily found (by summing the geometric series) that

$$Z_n=2(Z_0/2)^m, \tag{11}$$

where $m=5^n$ is the number of squares in the nth stage of construction. For a simple square the partition function is given by

$$Z_0=2(e^{4\beta J}+6+e^{-4\beta J}), \tag{12}$$

which can be (after a bit of elementary algebra) expressed as

$$Z_0=2^4\cosh^4(\beta J)[1+\tanh^4(\beta J)]. \tag{13}$$

Finally, substituting Eq. (13) into Eq. (11) we find

$$Z_n=2^{N_n}\cosh^M(\beta J)[1+\tanh^4(\beta J)]^m, \tag{14}$$

where $N_n=3m+1$ is the number of spins and $M=4m$ the number of bonds in the nth stage of construction of the fractal. Equation (14) represents the exact solution of the zero-field case, derived by Yang [5] using graph theory.

In the nonzero-field case, we could not obtain a single recursion relation for the full partition function and the closed-form expression in terms of elementary functions still seems unreachable. Nevertheless, as it will be shown in Sec. IV, Eqs. (3) and (4), with initial conditions (5), are sufficient for calculating the response functions at *any* stage of construction of the fractal with *arbitrary precision*. The thermodynamic limit (for a desired precision) is obtained when the response functions (calculated per spin) stop changing with increasing system size. We now proceed to describe the approach used to find the recursion relations for the second member of the CB fractal family.

III. RECURSION RELATIONS FOR THE CHECKERBOARD FRACTAL WITH $b=5$

As it was shown in Ref. [7], recursion relations for the partial partition functions in the general case of finitely ramified fractals, determined from the associated diagrams, can be expressed in the form

$$Z_i'=\sum_{k=1}^{n_i} d(k,i)e^{b(k,i)\beta H}\prod_{\ell=1}^{n_z} Z_\ell^{a(\ell,k,i)}. \tag{15}$$

Here $d(k,i)$ is the symmetry-induced degeneracy of individual graph contributions, with a total of $n_i\leqslant 2^{N_i}$ representative diagrams (here $N_i=32$ is the number of interior spins). The total number of associated diagrams is 2^{N_i}, so that $\Sigma_{k=1}^{n_i} d(k,i)=2^{N_i}$. With each representative diagram k we associate a product of n_c partial partition functions, corresponding to the n_c constituent structures of the previous stage of construction. The exponents $a(\ell,k,i)$ satisfy the condition $\Sigma_{\ell=1}^{n_z} a(\ell,k,i)=n_c$, where $n_z=6$ is the number of independent partial partition functions. Finally, the multiplicative factor $e^{b(k,i)\beta H}$ accounts for the fact that when the nth stage partial partition functions are multiplied, the field-dependent term in the Hamiltonian is taken into account as many times as there are nth stage structures connected by a given interior site. The quantity $b(k,i)$ is given by

$$b(k,i)=-\sum_{j=1}^{N_i} S_j(r_j-1), \tag{16}$$

where r_j is the number of nth stage structures joined in the $(n+1)$st stage by the jth interior site. For a more detailed description of this procedure, the reader is referred to Ref. [7].

From Eq. (15) it follows that the recursion relations are determined in full by the matrices $\hat{a}$, $\hat{b}$, and $\hat{d}$. To find the recursion relations for the second member of the checkerboard fractal family ($b=5$) with $N_i=32$ corresponding to $2^{N_i}=4\,294\,967\,296$ graphs, we have developed a devoted symbolic computer algorithm that performs the simple (but unfeasible for humans) task of graph counting. It requires 36 Mb of memory and runs approximately 30 h on a 133-MHz Pentium processor. Each of the obtained recursion relations has roughly 45 000 terms. Since the algorithm deals with integer arithmetics, the recursion relations are *exact*, but because of their size they are not presentable in a tabular form. The matrices $\hat{a}$, $\hat{b}$, and $\hat{d}$ in the form of data files, the original algorithm for their calculation as well as the algorithm for calculating the *numerically exact* response functions for arbitrary field and temperature, are obtainable from the authors upon request.

We have considered developing a more efficient algorithm based on the transfer matrix approach (see, e.g., Ref. [7]) for determining the recursion relations for the third member of the checkerboard family ($b=7$), but have declined to do so because of the fact that recursion relations are (at present) simply too big to be used for calculating the response functions. We put this off for the (hopefully near) future when computer hardware of considerably greater speed and memory becomes more readily available.

IV. THERMODYNAMIC RESPONSE FUNCTIONS

In this section we turn to calculating the response functions using the recursion relations for the partial partition functions. To avoid problems intrinsically associated with numeric differentiation, we deal directly with the *recursion relations for the field and temperature derivatives* of the partial partition functions. They are obtained by formal differentiation of Eq. (15), and are also in full determined by the coefficient matrices $\hat{a}$, $\hat{b}$, and $\hat{d}$. The explicit expressions for the recursion relations for the field and temperature derivatives are given by

$$\frac{\partial Z_i'}{\partial T}=\sum_{k=1}^{n_i} d(k,i)e^{\beta H b(k,i)}\left(\frac{-Hb(k,i)}{k_bT^2}+\sum_{\ell=1}^{n_z} a(\ell,k,i)\frac{1}{Z_\ell}\frac{\partial Z_\ell}{\partial T}\right)\prod_{\ell'=1}^{n_z} Z_{\ell'}^{a(\ell',k,i)}, \tag{17a}$$

$$\begin{aligned}\frac{\partial^2 Z_i'}{\partial T^2}=\sum_{k=1}^{n_i} d(k,i)e^{\beta H b(k,i)}\Bigg\{&\left(\frac{Hb(k,i)}{k_bT^2}\right)^2+\frac{2Hb(k,i)}{k_bT^3}\\ &+\sum_{\ell=1}^{n_z} a(\ell,k,i)\Bigg[-\frac{2Hb(k,i)}{k_bT^2}\frac{1}{Z_\ell}\frac{\partial Z_\ell}{\partial T}\\ &+[a(\ell,k,i)-1]\left(\frac{1}{Z_\ell}\frac{\partial Z_\ell}{\partial T}\right)^2+\frac{1}{Z_\ell}\frac{\partial^2 Z_\ell}{\partial T^2}\Bigg]\\ &+\sum_{\ell=1}^{n_z} a(\ell,k,i)\frac{1}{Z_\ell}\frac{\partial Z_\ell}{\partial T}\sum_{\ell'\neq\ell} a(\ell',k,i)\frac{1}{Z_{\ell'}}\frac{\partial Z_{\ell'}}{\partial T}\Bigg\}\\ &\times\prod_{\ell''=1}^{n_z} Z_{\ell''}^{a(\ell'',k,i)},\end{aligned} \tag{17b}$$

$$\frac{\partial Z_i'}{\partial H}=\sum_{k=1}^{n_i} d(k,i)e^{\beta H b(k,i)}\left(\beta b(k,i)+\sum_{\ell=1}^{n_z} a(\ell,k,i)\frac{1}{Z_\ell}\frac{\partial Z_\ell}{\partial H}\right)\prod_{\ell'=1}^{n_z} Z_{\ell'}^{a(\ell',k,i)}, \tag{17c}$$

$$\begin{aligned}\frac{\partial^2 Z_i'}{\partial H^2}=\sum_{k=1}^{n_i} d(k,i)e^{\beta H b(k,i)}\Bigg\{&\beta^2 b(k,i)^2+\sum_{\ell=1}^{n_z} a(\ell,k,i)\\ &\times\Bigg[2\beta b(k,i)\frac{1}{Z_\ell}\frac{\partial Z_\ell}{\partial H}+[a(\ell,k,i)-1]\left(\frac{1}{Z_\ell}\frac{\partial Z_\ell}{\partial H}\right)^2\\ &+\frac{1}{Z_\ell}\frac{\partial^2 Z_\ell}{\partial H^2}\Bigg]+\sum_{\ell=1}^{n_z} a(\ell,k,i)\frac{1}{Z_\ell}\frac{\partial Z_\ell}{\partial H}\\ &\times\sum_{\ell'\neq\ell} a(\ell',k,i)\frac{1}{Z_{\ell'}}\frac{\partial Z_{\ell'}}{\partial H}\Bigg\}\prod_{\ell''=1}^{n_z} Z_{\ell''}^{a(\ell'',k,i)}.\end{aligned} \tag{17d}$$

Starting from initial values (5) for the partial partition functions and the corresponding equations for their derivatives (for a given temperature and field) successive application of Eqs. (17) yields the derivatives for all higher stages of construction. The derivatives of the total partition function are then obtained as linear combinations of the partial derivatives [see Eq. (2)], and the specific heat, magnetization, and susceptibility per spin at the nth stage of fractal construction are obtained from the general formulas

$$C_H=\frac{k_BT^2}{N_n}\left[\frac{1}{Z}\frac{\partial^2 Z}{\partial T^2}+\frac{2}{ZT}\frac{\partial Z}{\partial T}-\left(\frac{1}{Z}\frac{\partial Z}{\partial T}\right)^2\right], \tag{18a}$$

$$\langle m\rangle=\frac{k_BT}{N_n}\frac{1}{Z}\frac{\partial Z}{\partial H}, \tag{18b}$$

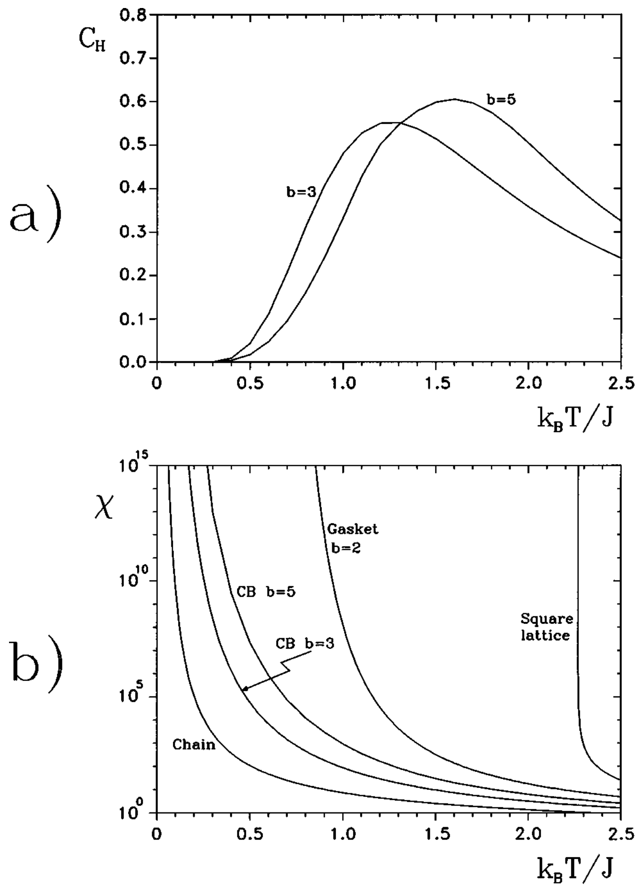

FIG. 4. (a) Specific heat and (b) susceptibility curves for the first two members of the checkerboard fractal family, with generator side lengths $b=3$ and $b=5$. The specific-heat curves display behavior reminiscent of finite-size systems. The susceptibility curves diverge in the zero-temperature limit, with slowing down of the correlation decay rate with increasing b. Curves for the Ising chain, Sierpiński gasket, and square lattice are shown for comparison.

$$\chi=\frac{k_BT}{N_n}\left[\frac{1}{Z}\,\frac{\partial^2 Z}{\partial H^2}-\left(\frac{1}{Z}\,\frac{\partial Z}{\partial H}\right)^2\right]. \tag{18c}$$

It should be stressed here that since we are dealing with *exact* recursion relations, the response functions can be obtained with *arbitrary* accuracy and are thus virtually *exact*. In Fig. 4 we present the results of our calculations for the specific heat and susceptibility for the first two members of the checkerboard fractal family. The calculated curves correspond to the thermodynamic limit in the sense that we have iterated beyond the point where the curves stop changing (within the chosen accuracy) with further steps of construction.

The specific heat shows only the regular Schottky peak, where both the intensity and the position of the maximum on the temperature scale increased with increasing b. The behavior of the specific-heat curves for the checkerboard fractal lattices with increasing b is thus reminiscent of the specific-heat behavior for increasing finite-size systems.

The susceptibility, on the other hand, shows divergent behavior in the zero-temperature limit, the temperature region where the susceptibility is ''practically divergent'' being increased for $b=5$ with respect to that for $b=3$. In the light of our previous results [7] for the Sierpínski gasket fractal family, we can expect these regions to monotonically increase with b and approach the critical temperature T_C of the square lattice in the limit $b\to\infty$. For comparison, in Fig. 4(b) we also show the results for the Sierpínski gasket with $b=2$ from Ref. [7], on the same scale, as well as the curves for the linear chain and the square lattice. It follows that the decay of correlations is much faster for both considered checkerboard fractals than that for the Sierpínski gasket, although the fractal dimension of the checkerboard fractal with $b=5$ ($d=\ln 13/\ln 5=1.5936$) is greater than that of the Sierpínski gasket with $b=2$ ($d=\ln 3/\ln 2=1.5849$). The decay of correlations is nevertheless slower than in the case of the chain, with a clear slowing down tendency with increasing b.

V. CONCLUSION

In this paper we find the exact recursion relations for the partition function, and its field and temperature derivatives, for the first two members of the checkerboard fractal family, in a nonzero external magnetic field. This represents *an exact general solution* of this model in a *nonzero external magnetic field*. The methodology for obtaining the exact recursion relations is rather involved, resulting in expressions that can only be stored in the form of computer data files. Therefrom, the *exact* specific-heat and susceptibility temperature dependence curves are calculated and compared with those pertinent to the Sierpínski gasket fractals, as well as to the corresponding Euclidean lattices. The crossover behavior seems to be much slower than in the case of the Sierpínski gaskets, implying that self-similarity, fractal dimension, and order of ramification are not the sole properties dictating the fractal to Euclidean crossover. In the special case of the zero-field Ising model of the first member ($b=3$), recursion relations are iterated to yield the exact closed-form expression for the partition function, in agreement with the result of Ref. [5].

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